Application of Moore products to temporal logics

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Computation Tree Logics

- We fix a rank type R with $0 \in R$.
- Σ: ranked alphabet of rank type R.
- T_Σ: terms over Σ (finite, no variables, ranked, ordered)
- CT_Σ: contexts over Σ.
- Formulae over Σ are generated by the grammar

$$\varphi \to p_\sigma, \text{ for all } \sigma \in \Sigma;$$
$$\varphi \to \neg\varphi \mid \varphi \vee \varphi \mid EF^+\varphi \mid EF^*\varphi$$
$$\varphi \to X_i\varphi, \text{ for all } i \in R.$$

A subset of the *CTL* modalities is allowed – a **fragment** of *CTL*.

Semantics reminder

Recall the semantics of some *CTL* modalities:

- $t \models p_\sigma$ iff $\mathrm{root}(t) = \sigma$;
- the Boolean connectives are treated as usual;
- $t \models X_i\varphi$ iff $t' \models \varphi$ for the ith **immediate** subtree t' of t;
- $t \models EF^+\varphi$ iff $t' \models \varphi$ for some **proper** subtree t' of t;
- $t \models EF^*\varphi$ iff $t' \models \varphi$ for **any** subtree t' of t.

A formula φ defines the tree language $L_\varphi = \{t \in T_\Sigma : t \models \varphi\}$.

Tree automata

Suppose Σ is a ranked alphabet.

- A Σ-algebra $\mathbb{A}$ consists of a nonempty **carrier** set A and a function $\sigma^{\mathbb{A}} : A^n \to A$ for each $n \in R$, $\sigma \in \Sigma_n$.
- Given $\mathbb{A}$, each term $t \in T_\Sigma$ **evaluates** to an element $t^{\mathbb{A}} \in A$.
- $\mathbb{A}$ is a **Σ-tree automaton** iff $A = \{t^{\mathbb{A}} : t \in T_\Sigma\}$.
- In any Σ-tree automaton $\mathbb{A}$ a context $\zeta \in CT_\Sigma$ induces a function $\zeta^{\mathbb{A}} : A \to A$.
- A tree language $L \subseteq T_\Sigma$ is **recognizable** by $\mathbb{A}$ if there is a set F of final states such that $L = \{t \in T_\Sigma : t^{\mathbb{A}} \in F\}$.
- A tree language is called **regular** iff it is recognizable by a finite tree automaton (that has a finite carrier set).

Automata over Bool

Recall that Bool is the ranked alphabet with $\mathrm{Bool}_n = \{\uparrow_n, \downarrow_n\}$ for each $n \in R$.

- The automaton $\mathbb{D}_0$ has the states $\{0, 1\}$. For each $n \in R$ we define $\uparrow_n^{\mathbb{D}_0}$ as the constant function with value 1, and $\downarrow_n^{\mathbb{D}_0}$ as the constant function with value 0.
- The automaton $\mathbb{E}_{EF*}$ also has the states $\{0, 1\}$. For each $n \in R$ we define $\uparrow_n^{\mathbb{E}_{EF*}}$ as the constant function with value 1, and $\downarrow_n^{\mathbb{E}_{EF*}}$ as the n-ary disjunction.
  Note that $\downarrow_0^{\mathbb{E}_{EF*}} = 0$, hence $\mathbb{E}_{EF*}$ is indeed a tree automaton.

Automata over Bool

Recall that Bool is the ranked alphabet with $\mathrm{Bool}_n = \{\uparrow_n, \downarrow_n\}$ for each $n \in R$.

- The automaton $\mathbb{E}_{EF^+}$ has the states $\{0, 1, 2\}$. For each $n \in R$ we define

$$\uparrow_n^{\mathbb{E}_{EF^+}}(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \forall i \ x_i = 0; \\ 2 & \text{otherwise} \end{cases}$$

  and

$$\downarrow_n^{\mathbb{E}_{EF^+}}(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } \forall i \ x_i = 0; \\ 2 & \text{otherwise.} \end{cases}$$

The cascade product

Suppose $\mathbb{A} = (A, \Sigma)$ and $\mathbb{B} = (B, \Delta)$ are tree automata and $\alpha = \{\alpha_n : n \in R\}$ is a family of functions where each α_n maps $A^n \times \Sigma_n$ to Δ_n.
Then the **cascade product** $\mathbb{A} \times_\alpha \mathbb{B}$ is the least subalgebra of $\mathbb{C} = (A \times B, \Sigma)$, where each $\sigma \in \Sigma_n$ is interpreted as

$$\sigma^{\mathbb{C}}\big((a_1, b_1), \dots, (a_n, b_n)\big) = (a, \delta^{\mathbb{B}}(b_1, \dots, b_n))$$

where $a = \sigma^{\mathbb{A}}(a_1, \dots, a_n)$ and $\delta = \alpha_n(a_1, \dots, a_n, \sigma)$.

The Moore product

Suppose $\mathbb{A} = (A, \Sigma)$ and $\mathbb{B} = (B, \Delta)$ are tree automata and $\beta : A \times \Sigma \to \Delta$ is a rank-preserving function.
Then the **Moore product** $\mathbb{A} \times_\beta \mathbb{B}$ is the least subalgebra of $\mathbb{C} = (A \times B, \Sigma)$, where each $\sigma \in \Sigma_n$ is interpreted as

$$\sigma^{\mathbb{C}}\big((a_1, b_1), \ldots, (a_n, b_n)\big) = (a, \delta^{\mathbb{B}}(b_1, \ldots, b_n))$$

where $a = \sigma^{\mathbb{A}}(a_1, \ldots, a_n)$ and $\delta = \beta(a, \sigma)$.

Varieties of finite tree automata

A nonempty class $\mathbf{V}$ if finite tree automata is called a (pseudo)**variety** iff it is closed under

- renamings;
- quotients (that is, taking homomorphic images);
- (finite) direct products.

If $\mathbf{V}$ is even closed under taking Moore (cascade, resp.) products, then $\mathbf{V}$ is called a Moore (cascade, resp.) variety.
If $\mathbf{V}$ is a class of finite tree automata, then $\langle \mathbf{V} \rangle_M$ denotes the least Moore variety $\mathbf{W}$ with $\mathbf{V} \subseteq \mathbf{W}$. The variety $\langle \mathbf{V} \rangle_c$ is defined similarly for the cascade product.

This talk

We will characterize the following varieties of tree automata:

- $\langle \mathbb{D}_0 \rangle_c$ (that corresponds to the logic $CTL(X)$);
- $\langle \mathbb{E}_{EF^+} \rangle_M$;
- $\langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_M$ (that corresponds to $CTL(EF^+)$);
- $\langle \mathbb{E}_{EF^*} \rangle_M$;
- $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_M$ (that corresponds to $CTL(EF^*)$);
- $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_c$ (that corresponds to $CTL(X + EF^+)$).

Note that $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_c = \langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_c$ holds, and the logics $CTL(X + EF^+)$ and $CTL(X + EF^*)$ are equivalent.

Definiteness and $\langle \mathbb{D}_0 \rangle_c$

- A tree automaton $\mathbb{A}$ is **definite** iff there exists an integer n such that $s^{\mathbb{A}} = t^{\mathbb{A}}$ holds whenever s and t are trees that "agree up to depth n".
- Definiteness is preserved under renamings, taking homomorphic images and cascade products. We call such a property a **cascade property**.
- **D** denotes the class of all definite tree automata.
- It clearly holds that $\langle \mathbb{D}_0 \rangle_c \subseteq \mathbf{D}$.

Characterization of $CTL(X)$

Theorem (Ésik). $\langle \mathbb{D}_0 \rangle_c = \mathbf{D}$.

Proof sketch. Call a congruence Θ of $\mathbb{A}$ **simple** if

- it collapses exactly two states and
- whenever $n \in R$, $\sigma \in \Sigma_n$ and $a_1 \Theta b_1, \ldots, a_n \Theta b_n$, it even holds that $\sigma^{\mathbb{A}}(a_1, \ldots, a_n) = \sigma^{\mathbb{A}}(b_1, \ldots, b_n)$.

Now we get the Theorem from

- if Θ is a simple congruence of $\mathbb{A}$, then $\mathbb{A}$ is a quotient of a cascade product $\mathbb{A}/\Theta \times_\alpha \mathbb{D}_0$;
- for any nontrivial definite tree automaton there exists a simple congruence.

Moore properties

A property $\mathcal{P}$ of tree automata is called a **Moore property** iff the class of all finite tree automata that satisfy $\mathcal{P}$ is a Moore variety.
Three Moore properties exactly characterize the variety $\langle \mathbb{E}_{EF^+} \rangle_M$:

- commutativity;
- monotonicity;
- maximal dependence.

Commutativity

A tree automaton $\mathbb{A}$ is **commutative** if for any arity $n \in R$, function symbol $\sigma \in \Sigma_n$, states $a_1, \ldots, a_n \in A$ and permutation π of $[n]$ it holds that

$$\sigma^{\mathbb{A}}(a_1, \ldots, a_n) = \sigma^{\mathbb{A}}(a_{\pi(1)}, \ldots, a_{\pi(n)}).$$

Commutativity is a Moore property.
Let **Com** denote the class of all commutative finite tree automata.
It holds that $\mathbb{E}_{EF^+}, \mathbb{D}_0 \in \textbf{Com}$.

Monotonicity

Let $\preceq_{\mathbb{A}}$ denote the accessibility relation of the tree automaton $\mathbb{A}$ (i.e. $a \preceq_{\mathbb{A}} b$ iff there exist a context $\zeta \in CT_{\Sigma}$ with $\zeta^{\mathbb{A}}(a) = b$).
Clearly, $\preceq_{\mathbb{A}}$ is a preorder for any $\mathbb{A}$.
If the accessibility relation of $\mathbb{A}$ is a partial order, we call $\mathbb{A}$ **monotone**.
Monotonicity is a cascade property. Let **Mon** denote the class of all monotone tree automata.
We have that $\mathbb{E}_{EF^{+}}$ is monotone (but, $\mathbb{D}_0$ is not).

Maximal dependency

We call a tree automaton $\mathbb{A}$ **maximal dependent** iff for any arity $n \in R$, function symbol $\sigma \in \Sigma_n$ and states $a_1, \ldots, a_{n-1}, b_1, b_2 \in A$ such that there exist indices $i, j \leq n-1$ with $b_1 \preceq_{\mathbb{A}} a_i$ and $b_2 \preceq_{\mathbb{A}} a_j$, then also

$$\sigma^{\mathbb{A}}(a_1, \ldots, a_{n-1}, b_1) = \sigma^{\mathbb{A}}(a_1, \ldots, a_{n-1}, b_2).$$

Maximal dependency is a Moore property; the corresponding Moore variety is denoted by **MaxDep**.
It is easy to check that $\mathbb{E}_{EF^+}, \mathbb{D}_0 \in$ **MaxDep**.

Characterizing $\langle \mathbb{E}_{EF^+} \rangle_M$

Theorem. $\langle \mathbb{E}_{EF^+} \rangle_M =$ **Com** $\cap$ **Mon** $\cap$ **MaxDep**.
Proof sketch. One direction is already proven.
For the other direction we can show that any nontrivial tree automaton $\mathbb{A} \in$ **Com** $\cap$ **Mon** $\cap$ **MaxDep** is. . .

- . . . either subdirectly reducible;
- . . . or there exists a proper congruence Θ of $\mathbb{A}$ such that $\mathbb{A}$ divides a Moore product $\mathbb{A}/\Theta \times_\beta \mathbb{F}$, for some $\mathbb{F} \in \langle \mathbb{E}_{EF^+} \rangle_M$.

This proves the Theorem.

The difference between $\mathbb{E}_{EF^*}$ and $\mathbb{E}_{EF^+}$

Call a tree automaton $\mathbb{A}$ **stutter invariant** iff for all arity $n \in R$, function symbol $\sigma \in \Sigma_n$ and states $a_1, \ldots, a_n \in A$ it holds that

$$\sigma^{\mathbb{A}}(a_1, \ldots, a_n) = \sigma^{\mathbb{A}}(a_1, \ldots, a_{n-1}, \sigma^{\mathbb{A}}(a_1, \ldots, a_n)).$$

Stutter invariance is a Moore property. Let **Stu** denote the corresponding Moore variety.

$\mathbb{E}_{EF^*}$ and $\mathbb{D}_0$ are contained in **Stu**. However, $\mathbb{E}_{EF^+}$ is not.

Characterizing $\langle \mathbb{E}_{EF^*} \rangle_M$

Theorem. $\langle \mathbb{E}_{EF^*} \rangle_M = \mathbf{Com} \cap \mathbf{Mon} \cap \mathbf{MaxDep} \cap \mathbf{Stu}$.
Proof sketch. The proof is similar to the case of strict EF, altough the construction is slightly more complicated.

This talk – reminder

We will characterize the following varieties of tree automata:

- $\langle \mathbb{D}_0 \rangle_c$ (that corresponds to the logic $CTL(X)$);
- $\langle \mathbb{E}_{EF^+} \rangle_M$;
- $\langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_M$ (that corresponds to $CTL(EF^+)$);
- $\langle \mathbb{E}_{EF^*} \rangle_M$;
- $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_M$ (that corresponds to $CTL(EF^*)$);
- $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_c$ (that corresponds to $CTL(X + EF^+)$).

Handling $\mathbb{D}_0$

Let $\mathbf{D}_0$ denote the (decidable) Moore variety $\langle \mathbb{D}_0 \rangle_M$.
Lemma. For any variety $\mathbf{V}$ it holds that

$$\langle \mathbf{V} \cup \{\mathbb{D}_0\} \rangle_M = \langle \mathbf{V} \rangle_M \times \mathbf{D}_0.$$

Proof sketch.

- Any Moore product $\mathbb{A} \times_\beta \mathbb{D}$ with $\mathbb{D} \in \mathbf{D}_0$ is a quotient of some **direct** product $\mathbb{A} \times \mathbb{D}'$, with $\mathbb{D}' \in \mathbf{D}_0$.
- Any Moore product $\mathbb{D} \times_\beta \mathbb{A}$ with $\mathbb{D} \in \mathbf{D}_0$ is isomorphic to some **direct** product $\mathbb{A}' \times \mathbb{D}$, where $\mathbb{A}'$ is a renaming of $\mathbb{A}$.

This proves the Lemma.

Aiming **Mon** $\times$ **D**$_0$

From the two characterization theorems and the previous lemma we get the following:

$$\langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_M = \mathbf{Com} \cap (\mathbf{Mon} \times \mathbf{D}_0) \cap \mathbf{MaxDep};$$
$$\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_M = \mathbf{Com} \cap (\mathbf{Mon} \times \mathbf{D}_0) \cap \mathbf{MaxDep} \cap \mathbf{Stu}.$$

Altough this is already a structural characterization, it is not readily decidable.

Component dependency

Let $\approx_{\mathbb{A}}$ denote the equivalence relation

$$a \approx_{\mathbb{A}} b \Leftrightarrow a \preceq_{\mathbb{A}} b \wedge b \preceq_{\mathbb{A}} a.$$

An automaton $\mathbb{A}$ is **component dependent** if for each arity $n \in R$, $\sigma \in \Sigma_n$ and $a_1 \approx_{\mathbb{A}} b_1, \ldots, a_n \approx_{\mathbb{A}} b_n$ it holds that

$$\sigma^{\mathbb{A}}(a_1, \ldots, a_n) = \sigma^{\mathbb{A}}(b_1, \ldots, b_n).$$

Component dependency is a Moore property; **CompDep** denotes the corresponding variety of finite tree automata.
Note that $\mathbb{D}_0 \in$ **CompDep**, and of course **Mon** $\subseteq$ **CompDep** holds.

Componentwise uniqueness

Suppose for a Σ-tree automaton $\mathbb{A}$ that whenever $a, b \in A$ are states and $\zeta, \xi \in CT_\Sigma$ are contexts such that

- $\zeta^{\mathbb{A}}(a) = b$
- $\xi^{\mathbb{A}}(b) = a$
- and $\mathrm{Root}(\zeta) = \mathrm{Root}(\xi)$

then $a = b$ has to hold.

Then we call $\mathbb{A}$ a **componentwise unique** automaton.

Componentwise uniqueness is a Moore property. **CWU** denotes the corresponding Moore variety.

It is easy to check that $\mathbb{D}_0 \in$ **CWU** and **Mon** $\subseteq$ **CWU**.

Deciding $\mathbf{Mon} \times \mathbf{D}_0$

Theorem. $\mathbf{Mon} \times \mathbf{D}_0 = \mathbf{CompDep} \cap \mathbf{CWU}$.

Proof sketch. One direction is clear.

The other direction comes from the following facts:

- If $\mathbb{A}$ is component dependent, then $\approx_{\mathbb{A}}$ is a congruence.
- If $\approx_{\mathbb{A}}$ is a congruence, then $\mathbb{A}/\approx_{\mathbb{A}}$ is monotone.
- If $\mathbb{A}$ is componentwise unique and component dependent, then $\mathbb{A}$ is a quotient of a direct product $\mathbb{A}/\approx_{\mathbb{A}} \times \mathbb{D}$, with $\mathbb{D} \in \mathbf{D}_0$.

Now this gives us a decidability result.

Decidable *CTL* fragments

Theorem.

$$\langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_M = \mathbf{Com} \cap \mathbf{CompDep} \cap \mathbf{CWU} \cap \mathbf{MaxDep};$$
$$\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_M = \langle \mathbb{E}_{EF^+}, \mathbb{D}_0 \rangle_M \cap \mathbf{Stu}.$$

Since all five Moore properties involved in the characterization above is decidable (even in polynomial time), membership for these varieties (hence, definability in the logics $CTL(EF^+)$ and $CTL(EF^*)$) is decidable.

The fragment $CTL(X + EF)$

Suppose $\mathbb{A}$ is a tree automaton and $k > 0$ is an integer such that whenever $n \geq 0$ and

- $t \in T_\Sigma(X_n)$ having all variable-labeled leaves x_i in depth at least k;
- $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ are states with $a_i, b_i \approx_{\mathbb{A}} a_1$ for all $i \in [n]$;
- $t^{\mathbb{A}}(a_1, \ldots, a_n) \approx_{\mathbb{A}} t^{\mathbb{A}}(b_1, \ldots, b_n) \approx_{\mathbb{A}} a_1$;

then even $t^{\mathbb{A}}(a_1, \ldots, a_n) = t^{\mathbb{A}}(b_1, \ldots, b_n)$ holds.
Then $\mathbb{A}$ is called an XF-automaton.

The fragment $CTL(X + EF)$

The class of all finite XF-automata, denoted **XF**, is a cascade variety and contains both $\mathbb{D}_0$ and $\mathbb{E}_{EF^*}$.
Moreover, it can be shown that
Theorem (Ésik). $\langle \mathbb{E}_{EF^*}, \mathbb{D}_0 \rangle_c = \mathbf{XF}$.
This again gives us a decidable characterization.

Summary

We gave examples how the concepts of Moore and cascade varieties can be used to show decidability of a given fragment of the logic CTL. Namely, the following fragments are known to be decidable so far:

- $CTL(X)$;
- $CTL(EF^*)$;
- $CTL(EF^+)$;
- $CTL(X + EF)$.

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